

# Scattering of internal waves from small sea bottom inhomogeneities

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## Abstract

The problem of scattering of linear internal waves from small compact sea bottom inhomogeneities is considered from the point of view of mode-to-mode scattering. A simple formula for modal conversion coefficients  $C_{nm}$ , quantifying the amount of energy that is scattered into the  $n$ -th mode from the incident field  $m$ -th mode, is derived. In this formula the representation of inhomogeneities by their expansions into the Fourier and Fourier-Bessel series with respect to angular and radial coordinates respectively are used. Results of calculations, performed in a simple model case, are presented. The obtained formula can be used for a formulation of the inverse problem, as it was done in the acoustic case [2, 3].

Keywords: internal wave, scattering

## 1 Introduction

The concept of mode-to-mode scattering was considered in the context of the acoustic scattering from small compact irregularities of the ocean floor by Wetton, Fawcett [1]. In their work some simple formulas for modal conversion coefficients, quantifying the amount of energy that is scattered from one normal mode of the sound field to another, were derived. Recently new formulas for these coefficients were obtained by Zakharenko [2] and applied to the inverse scattering problem in the subsequent work [3]. This paper contains the detailed derivation of such formulas in the case of scattering of linear internal waves from small compact sea bottom inhomogeneities. Some numerical examples are presented.

## 2 Formulation and derivation of the main result

We shall use the linearized equations for inviscid, incompressible stably stratified fluid, written for the harmonic dependence on the time with the factor  $e^{-i\omega t}$  in the form

$$\begin{aligned} -i\omega\rho_0 u + \beta P_x &= 0, \\ -i\omega\rho_0 v + \beta P_y &= 0, \\ -i\omega\rho_0 w + \beta P_z + \beta\rho_1 &= 0, \\ -i\omega\rho_1 + w\rho_{0z} &= 0, \\ u_x + v_y + w_z &= 0, \end{aligned} \tag{1}$$

where  $x$ ,  $y$ , and  $z$  are the Cartesian co-ordinates with the  $z$ -axis directed upward,  $\rho_0 = \rho_0(z)$  is the undisturbed density,  $\rho_1 = \rho_1(x, y, z)$  is the perturbation of density due to motion,  $P$  is the pressure, and  $u$ ,  $v$ ,  $w$  are the  $x$ ,  $y$  and  $z$  components of velocity respectively. The variables are nondimensional, based on a length scale  $\bar{h}$  (a typical vertical dimension), a time scale  $\bar{N}^{-1}$  (where  $\bar{N}$  is a typical value of the Brunt-Väisälä frequency), and a density scale  $\bar{\rho}$  (a typical value of the density). The parameter  $\beta$  is  $g/(\bar{h}\bar{N}^2)$ , where  $g$  is the gravity acceleration.

The boundary conditions for these equations are

$$\begin{aligned} w &= 0 \quad \text{at} \quad z = 0, \\ w &= -uH_x - vH_y \quad \text{at} \quad z = -H, \end{aligned} \tag{2}$$

where  $H = H(x, y)$  is the bottom topography.

We introduce a small parameter  $\epsilon$ , and postulate that the components of velocity and the pressure are represented in the form

$$u = u_0 + \epsilon u_1 + \dots, \quad v = v_0 + \epsilon v_1 + \dots,$$

$$w = w_0 + \epsilon w_1 + \dots, \quad P = P_0 + \epsilon P_1 + \dots$$

We suppose also that the bottom topography is represented in the form  $H = h_0 + \epsilon h_1$ , where  $h_0$  is constant and  $h_1 = h_1(x, y)$  is a function of  $x$ ,  $y$  vanishing outside the bounded domain  $\Omega$ , which in the sequel is called a domain of inhomogeneity.

Excluding from the system (1)  $\rho_1$  and substituting the introduced expansions, we obtain

$$\begin{aligned}
-i\omega\rho_0(u_0 + \epsilon u_1 + \dots) + \beta(P_{0x} + \epsilon P_{1x} + \dots) &= 0, \\
-i\omega\rho_0(v_0 + \epsilon v_1 + \dots) + \beta(P_{0y} + \epsilon P_{1y} + \dots) &= 0, \\
(\omega^2\rho_0 + \beta\rho_{0z})(w_0 + \epsilon w_1 + \dots) + i\omega\beta(P_{0z} + \epsilon P_{1z} + \dots) &= 0, \\
(u_{0x} + \epsilon u_{1x} + \dots) + (v_{0y} + \epsilon v_{1y} + \dots) + w_{0z} + \epsilon w_{1z} + \dots &= 0,
\end{aligned} \tag{3}$$

with the boundary conditions

$$\begin{aligned}
w_0 + \epsilon w_1 + \dots &= 0 \quad \text{at } z = 0, \\
w_0 + \epsilon w_1 + \dots &= -(u_0 + \epsilon u_1 + \dots)(h_{x0} + \epsilon h_{1x}) \\
&\quad - \epsilon(v_0 + \epsilon v_1 + \dots)(h_{1y} + \dots) \quad \text{at } z = -H.
\end{aligned} \tag{4}$$

Separating terms in various orders of  $\epsilon$ , we obtain a sequence of boundary problems.

At order  $O(\epsilon^0)$  we have

$$\begin{aligned}
-i\omega\rho_0 u_0 + \beta P_{0x} &= 0, \\
-i\omega\rho_0 v_0 + \beta P_{0y} &= 0, \\
(\omega^2\rho_0 + \beta\rho_{0z})w_0 + i\omega\beta P_{0z} &= 0, \\
u_{0x} + v_{0y} + w_{0z} &= 0,
\end{aligned} \tag{5}$$

with the boundary conditions

$$\begin{aligned}
w_0 &= 0 \quad \text{at } z = 0 \\
w_0 &= 0 \quad \text{at } z = -h_0.
\end{aligned}$$

Differentiating the third equation in (5) twice with respect to  $x$  and twice with respect to  $y$ , summing obtained equations and replacing  $\beta(P_{0zxx} + P_{0zyy})$  by  $-i\omega(\rho_0 w_{0z})_z$ , we obtain

$$(\omega^2\rho_0 + \beta\rho_{0z})(w_{0xx} + w_{0yy}) + \omega^2(\rho_0 w_{0z})_z = 0. \tag{6}$$

We seek a solution to this equation in the form of the sum of normal modes  $w_0 = e^{i(kx+ly)}\phi(z)$ , where  $\phi$  is the eigenfunction of the spectral boundary problem

$$\begin{aligned}
-(\omega^2\rho_0 + \beta\rho_{0z})(k^2 + l^2)\phi + \omega^2(\rho_0\phi_z)_z &= 0, \\
\phi(0) = \phi(-h_0) &= 0,
\end{aligned} \tag{7}$$

with the eigenvalue  $\lambda = k^2 + l^2$ . It is well known that the problem (7) has countably many eigenvalues  $\lambda_n$ , which are all positive. The corresponding real eigenfunctions  $\phi_n$  we normalize by the condition

$$-\int_{-h_0}^0 (\omega^2\rho_0 + \beta\rho_{0z})\phi^2 dz = \frac{\omega^2}{k^2 + l^2} \int_{-h_0}^0 \rho_0(\phi_z)^2 dz = 1. \tag{8}$$

The eigenfunctions  $\phi_n$  and  $\phi_m$  with  $n \neq m$  are also orthogonal

$$(\phi_n, \phi_m) = 0 \quad (9)$$

with respect to the inner product

$$(\phi, \psi) = - \int_{-h_0}^0 (\omega^2 \rho_0 + \beta \rho_{0z}) \phi \psi dz \quad (10)$$

In our scattering problem  $w_0$  is the incident field, and we shall calculate the main term of scattering field  $w_1$ , so we act in the framework of the Born approximation.

At the first order of  $\epsilon$  we obtain the following system of equations:

$$\begin{aligned} -i\omega \rho_0 u_1 + \beta P_{1x} &= 0, \\ -i\omega \rho_0 v_1 + \beta P_{1y} &= 0, \\ (\omega^2 \rho_0 + \beta \rho_{0z}) w_1 + i\omega \beta P_{1z} &= 0, \\ u_{1x} + v_{1y} + w_{1z} &= 0, \end{aligned} \quad (11)$$

with the boundary conditions

$$\begin{aligned} w_1 &= 0 \quad \text{at} \quad z = 0, \\ w_1 &= -u_0 h_{1x} - v_0 h_{1y} \quad \text{at} \quad z = -h_0 - \epsilon h_1. \end{aligned} \quad (12)$$

So far as we are interesting in the connection of modal contents of incident and scattering fields, we suppose that the incident field consists of one mode  $w_0 = e^{i(k_n x + l_n y)} \phi_n(z)$ . Reducing the second boundary condition (12) to the boundary  $z = -h_0$  with taking into account the explicit form of  $w_0$ , we obtain the new boundary condition for  $w_1$  at the boundary  $z = -h_0$ :

$$w_1 = \left( h_1 - \frac{ik_n}{k_n^2 + l_n^2} h_{1x} - \frac{il_n}{k_n^2 + l_n^2} h_{1y} \right) e^{i(k_n x + l_n y)} \phi_{nz}. \quad (13)$$

Reducing the system (11) in the same manner as it was done for the system (5), we obtain the equation for  $w_1$ :

$$(\omega^2 \rho_0 + \beta \rho_{0z})(w_{1xx} + w_{1yy}) + \omega^2(\rho_0 w_{1z})_z = 0. \quad (14)$$

We seek the scattering field in the form  $w_1 = \sum_{m=1}^N C_{nm}(x, y) \phi_m$ , the functions  $C_{nm}(x, y)$  are called the modal conversion coefficients. To obtain the equation for  $C_{nm}$  we substitute the postulated form of  $w_1$  to the (13), multiply it by the function  $\phi_m$  and integrate from  $-h_0$  to 0. Using the conditions of orthogonality and normalization (8), (9) and the boundary condition (13), we finally obtain

$$\frac{\partial^2}{\partial x^2} C_{nm} + \frac{\partial^2}{\partial y^2} C_{nm} + (k_m^2 + l_m^2) C_{nm} = F, \quad (15)$$

where

$$F = \omega^2 \rho_0 \left( h_1 - \frac{ik_n}{k_n^2 + l_n^2} h_{1x} - \frac{il_n}{k_n^2 + l_n^2} h_{1y} \right) e^{i(k_n x + l_n y)} \phi_{nz}(-h_0) \phi_{mz}(-h_0).$$

Writing the solution to the equation (15) as the convolution of the fundamental solution (Green function) of the Helmholtz operator  $G = (-i/4)H_0^{(1)}(\sqrt{k_m^2 + l_m^2}R)$  with the right-hand side  $F$ , we have

$$C_{nm}(x_r, y_r) = -\frac{i}{4} \int_x \int_y F H_0^{(1)}(\sqrt{k_m^2 + l_m^2}R) dy dx, \quad (16)$$

where  $R = \sqrt{(x - x_r)^2 + (y - y_r)^2}$  and by the index  $r$  we designate the point of registration of the field.

Integrating by parts the terms containing  $h_{1x}, h_{1y}$  and passing to the cylindrical coordinate system with the origin in our domain of inhomogeneity and such that  $k_n = \kappa_n$ ,  $l_n = 0$ ,  $x = r \cos \alpha$ ,  $y = r \sin \alpha$ , we obtain

$$C_{nm} = -\frac{1}{4} \frac{\kappa_m}{\kappa_n} G \int_0^\infty \int_0^{2\pi} h_1 e^{i\kappa_n r \cos \alpha} \cos(\psi - \alpha_r) H_1^{(1)}(\kappa_m R) r d\alpha dr, \quad (17)$$

$G = \omega^2 \rho_0 \phi_{nz}(-h_0) \phi_{mz}(-h_0)$ ,  $R = \sqrt{r^2 + r_r^2 - 2rr_r \cos(\alpha - \alpha_r)}$ ,  $(r_r, \alpha_r)$  are the polar coordinates of the registration point,  $\tan(\psi) = r \sin(\alpha - \alpha_r) / (r_r - r \cos(\alpha - \alpha_r))$ .

Using the addition theorem for the Bessel functions we express contained in (17)  $\cos \psi H_1^{(1)}(\kappa_m R)$  and  $\sin \psi H_1^{(1)}(\kappa_m R)$  in the form:

$$\begin{Bmatrix} \cos(\psi) \\ \sin(\psi) \end{Bmatrix} H_1^{(1)}(\kappa_m R) = \sum_{k=-\infty}^{\infty} H_{k+1}^{(1)}(\kappa_m r_r) J_k(\kappa_m r) \begin{Bmatrix} \cos k(\alpha - \alpha_r) \\ \sin k(\alpha - \alpha_r) \end{Bmatrix}.$$

From now on we shall assume that the distance  $r_r$  to the registration point is big enough to replace the functions  $H_{k+1}^{(1)}(\kappa_m r_r)$  by their asymptotics

$$H_{k+1}^{(1)}(\kappa_m r_r) \approx \sqrt{2/(\pi \kappa_m r_r)} \exp[i(\kappa_m r_r - (\pi/2)(k+1) - \pi/4)].$$

Then, expanding  $h_1(r, \alpha)$  as function of  $\alpha$  in Fourier series with the coefficients  $\tilde{h}_{1\nu}(r)$ , after integration with respect to  $\alpha$ , we obtain

$$\begin{aligned}
C_{nm} = & \frac{i\sqrt{2\pi}}{2} \frac{\sqrt{\kappa_m} \exp(i\kappa_m r_r - i\pi/4)}{\kappa_n \sqrt{r_r}} G \cos \alpha_r \sum_{\nu=-\infty}^{\infty} (i)^\nu e^{-i\nu\alpha_0} \\
& \times \sum_{k=-\infty}^{\infty} e^{-ik\alpha_r} \int_0^{\infty} \tilde{h}_{1\nu}(r) J_k(\kappa_m r) J_{\nu+k}(\kappa_n r) r dr
\end{aligned} \tag{18}$$

Changing the order of integration and summation we can achieve further simplification by using the formula

$$\sum_{k=-\infty}^{\infty} J_k(\kappa_m r) J_{\nu+k}(\kappa_n r) e^{-ik\alpha_r} = J_\nu(\xi r) e^{-i\nu\theta},$$

where  $\xi = \sqrt{\kappa_m^2 + \kappa_n^2 - 2\kappa_m \kappa_n \cos \alpha_r}$ ,  $\theta = \arctan \frac{\kappa_m \sin \alpha_r}{\kappa_n - \kappa_m \cos \alpha_r}$ . We expand now the radial coefficients  $\tilde{h}_{1\nu}(r)$  on the segment  $[0, L]$ , where they do not vanish, in the Fourier-Bessel series

$$\tilde{h}_{1\nu}(r) = \sum_{p=1}^{\infty} f_p^\nu J_\nu \left( \frac{\gamma_p^\nu}{L} r \right),$$

where  $\gamma_p^\nu$  are the positive roots of the function  $J_\nu$ ,  $J_\nu(\gamma_p^\nu) = 0$ . Substituting this expansion in (18) and taking into account that

$$\int_0^L J_\nu \left( \frac{\gamma_p^\nu}{L} r \right) J_\nu(\xi r) r dr = \frac{-L^2 \gamma_p^\nu J_\nu(\xi L) J'_\nu(\gamma_p^\nu)}{\gamma_p^{\nu^2} - \xi^2 L^2},$$

we obtain the final expression for modal conversion coefficients

$$\begin{aligned}
C_{nm} = & - \frac{iL^2 \sqrt{2\pi}}{2} \frac{\sqrt{\kappa_m} \exp(i\kappa_m r_r - i\pi/4)}{\kappa_n \sqrt{r_r}} G \cos \alpha_r \\
& \times \sum_{\nu=-\infty}^{\infty} (i)^\nu J_\nu(\xi L) e^{-i\nu(\alpha_0 + \theta)} \sum_{p=1}^{\infty} f_p^\nu \frac{\gamma_p^\nu J'_\nu(\gamma_p^\nu)}{\gamma_p^{\nu^2} - \xi^2 L^2}.
\end{aligned} \tag{19}$$

### 3 Numerical examples

For a model example we choose  $\rho = e^{-\lambda z}$ ,  $\beta = \lambda^{-1}$  and  $H = 1$ . Then the spectral boundary problem is written in the form

$$\begin{aligned}\omega^2 \phi_{zz} - \omega^2 \lambda \phi_z - \kappa^2 (\omega^2 - 1) \phi &= 0, \\ \phi(0) &= 0, \quad \phi(-1) = 0.\end{aligned}$$

The eigenfunctions of such a problem are  $\phi = A e^{\lambda z/2} \sin((l+1)\pi z)$  with the eigenvalues

$$\kappa = \frac{\omega \sqrt{(l+1)^2 \pi^2 + \lambda^2/4}}{\sqrt{1-\omega^2}}.$$

Here  $A = \sqrt{2}/(\sqrt{1-\omega^2})$  by the condition (9). For the calculations the value of parameter  $\lambda$  was taken to be equal to 0.003, which corresponds to the typical stratification in the ocean shelf zones. The domain of inhomogeneity has the form of the ellipse with the big and small radii  $a$  and  $b$  of which were taken in proportion  $a : b = 2 : 1$ , and in this region

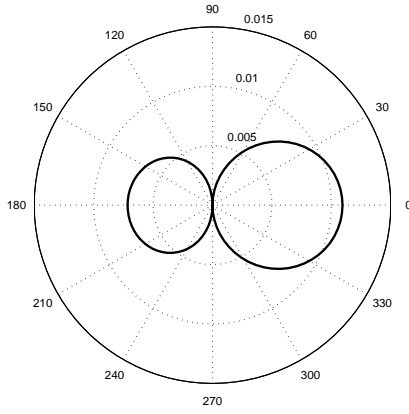
$$h_1(x, y) = 0.05 \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

In the figure are presented the results of calculations with  $\omega = 0.5$  and the angle of incident field  $\alpha_0 = 0$ , conducted for various wave sizes  $\kappa a$  of the scatterer. We note that according to the meaning of the small parameter  $\epsilon$ , in these calculations  $\epsilon = 0.05$ . For the presentation of results we use the scattering amplitude

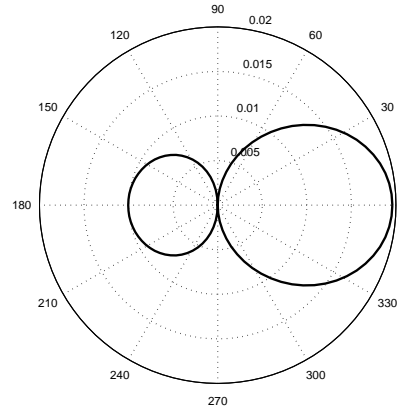
$$F_{nm}(\alpha_r) = \left( \frac{e^{i\kappa_m r_r}}{\sqrt{r_r}} \right)^{-1} C_{nm}(\alpha_r).$$

### References

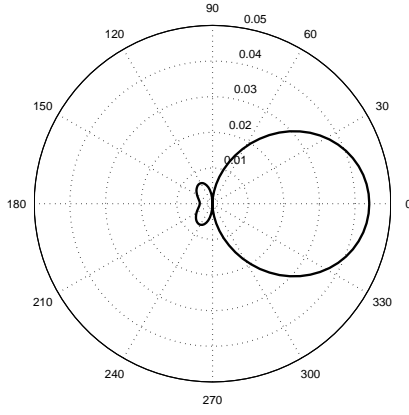
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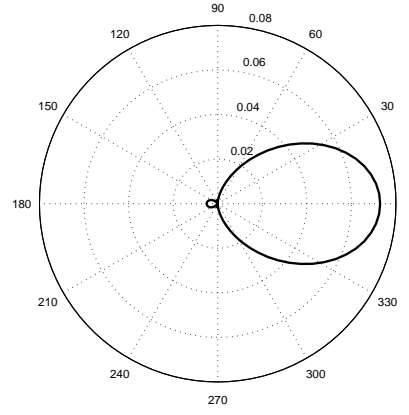
(a)  $|F_{11}|$



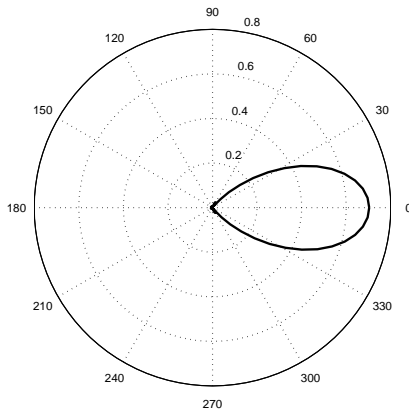
(b)  $|F_{12}|$



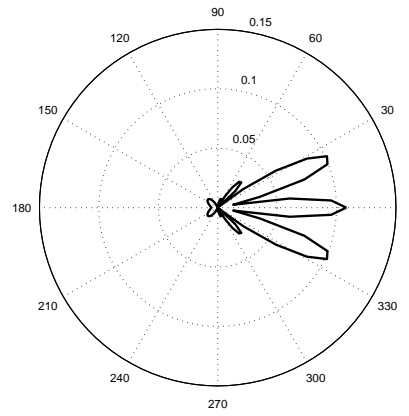
(c)  $|F_{11}|$



(d)  $|F_{12}|$



(e)  $|F_{11}|$



(f)  $|F_{12}|$

Figure 1: Absolute value of scattering amplitude:  $\kappa a = 1$  (a,b),  $\kappa a = 2$  (c,d),  $\kappa a = 8$  (f,g)